JOURNAL OF APPROXIMATION THEORY 69, 250–268 (1992)

# On the Distribution of Simple Zeros of Polynomials

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*Communicated by Vilmos Totik*

Received February 8, 1991; revised March 1, 1991

DEDICATED TO THE MEMORY OF LOTHAR COLLATZ

Erdős and Turán discussed in (*Ann. of Math.* **41** (1940), 162–173; **51** (1950), 105–119) the distribution of the zeros of monic polynomials if their Chebyshev norm on $[-1, 1]$ or on the unit disk is known. We sharpen this result to the case that all zeros of the polynomials are simple. As applications, estimates for the distribution of the zeros of orthogonal polynomials and the distribution of the alternation points in Chebyshev polynomial approximation are given. This last result sharpens a well-known error bound of Kadec (*Amer. Math. Soc. Transl.* **26** (1963), 231–234). 

## 1. INTRODUCTION AND MAIN THEOREMS

In [3] Erdős and Turán considered the distribution of the zeros of a monic polynomial $p_n \in \Pi_n$, where $\Pi_n$ denotes the set of all algebraic polynomials of degree at most $n$. To be precise, we associate with $p_n$ the zero counting measure

$$\tau_n(A) = \frac{\text{number of zeros of } p_n \text{ on } A}{n}, \tag{1.1}$$

where $A$ is any point set of $\mathbb{C}$. Let $\mu$ be the equilibrium distribution of $[-1, 1]$, i.e.,

$$\mu(A) = \frac{1}{\pi} \int_\alpha^\beta \frac{dx}{\sqrt{1-x^2}} \tag{1.2}$$

for any subinterval $A = [\alpha, \beta]$ of $[-1, 1]$ and let us assume that all zeros of $p_n$ lie in $[-1, 1]$. Then Erdős and Turán proved that

$$|(\tau_n - \mu)([\alpha, \beta])| \leq \frac{8}{\log 3} \sqrt{\frac{\log A_n}{n}} \tag{1.3}$$



0021-9045/92 $5.00

for any interval $[\alpha, \beta] \subset [-1, 1]$, where

$$\max_{-1 \leqslant x \leqslant 1} |p_n(x)| \leqslant A_n \frac{1}{2^n}. \tag{1.4}$$

This result is sharp up to the constant 8/log 3. But if we know that all zeros of $p_n$ are simple then the estimate can be strengthened.

Let us henceforth assume that all zeros $x_i$ of $p_n$ are simple and contained in $[-1, 1]$, i.e.,

$$-1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant 1. \tag{1.5}$$

Moreover, we assume a lower bound for the derivative $|p_n'(x_i)|$, namely

$$|p_n'(x_i)| \geqslant \frac{1}{B_n} \frac{1}{2^n}, \qquad 1 \leqslant i \leqslant n. \tag{1.6}$$

Then we can formulate our main result as

THEOREM A. *Let $p_n$ be a monic polynomial with zeros* (1.5) *satisfying the conditions* (1.4) *and* (1.6), $n \geqslant 2$. *Then there exists a constant $c$ (independent of $n$) such that*

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant c \frac{\log C_n}{n} \log n \tag{1.7}$$

*for any interval* $[\alpha, \beta] \subset [-1, 1]$, *where*

$$C_n = \max(A_n, B_n, n).$$

We want to formulate Theorem A in a potential theoretic way, which has the advantage that the conditions on $p_n(x)$ are more symmetric and already give some insight into the method of our later proof.

Let $G(z)$ denote Green's function of $\bar{\mathbb{C}} \setminus [-1, 1]$ with pole at infinity, i.e., $G(z) = \log |z + \sqrt{z^2 - 1}|$, where the function $\sqrt{z^2 - 1}/z$ is 1 at infinity. Bernstein's inequality together with (1.4) yields

$$\frac{1}{n} \log |p_n(z)| - G(z) - \log \frac{1}{2} \leqslant \frac{\log A_n}{n} \qquad \text{for all} \quad z \in \mathbb{C}. \tag{1.8}$$

The interpolation formula of Lagrange shows that

$$1 = \sum_{i=1}^{n} \frac{p_n(z)}{p_n'(x_i)(z - x_i)}.$$

For $z \notin [-1, 1]$, let $d(z)$ denote the distance of the point $z$ to the interval $[-1, 1]$. Then, using (1.6),

$$1 \leqslant n \frac{|p_n(z)|}{d(z)} B_n 2^n$$

or

$$|p_n(z)| \geqslant \frac{1}{n} \frac{d(z)}{B_n} \frac{1}{2^n}. \tag{1.9}$$

For $z \in \Gamma_\sigma$, where $\sigma > 1$ and $\Gamma_\sigma = \{z \in \mathbb{C}\colon G(z) = \log \sigma\}$ is a level line of the Green's function $G(z)$, we have

$$\min_{z \in \Gamma_\sigma} d(z) = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right) - 1$$

since $\Gamma_\sigma$ is an ellipse with foci $+1$ and $-1$ and major axis $\sigma + 1/\sigma$. If we choose

$$\sigma = \sigma_n := 1 + n^{-8} \tag{1.10}$$

the inequality (1.9) leads to

$$\frac{1}{n} \log |p_n(z)| - G(z) - \log \frac{1}{2} \geqslant -\kappa \frac{\log C_n}{n} \tag{1.11}$$

for $z \in \Gamma_{\sigma_n}$, where $\kappa > 0$ is an absolute constant independent of $n$. The minimum principle for harmonic functions shows that (1.11) is satisfied for all $z$ with $G(z) \geqslant \log \sigma_n$. Summarizing (1.8) and (1.11) we get

$$\left| \frac{1}{n} \log |p_n(z)| - G(z) - \log \frac{1}{2} \right| \leqslant \kappa \frac{\log C_n}{n} \tag{1.12}$$

for all $z$, where $G(z) \geqslant \log \sigma_n$.

Since $-(1/n) \log |p_n(z)|$ is the logarithmic potential $U^{\tau_n}$ of the measure $\tau_n$ and $U^\mu(z) = -G(z) - \log \frac{1}{2}$ is the logarithmic potential of the equilibrium distribution $\mu$, (1.12) can be written as

$$|U^{\tau_n}(z) - U^\mu(z)| \leqslant \kappa \frac{\log C_n}{n} \tag{1.13}$$

for all $z$ with $G(z) \geqslant \log \sigma_n$.

THEOREM B. *Let $p_n \in \Pi_n$, $n \geqslant 2$ be a monic polynomial with simple zeros* *in* $[-1, 1]$ *such that for the zero counting measure* $\tau_n$ *and the equilibrium measure* $\mu$ *of* $[-1, 1]$ *the inequality* (1.13) *is satisfied. Then there exists a constant* $c > 0$ *(independent of* $n$*) such that* (1.7) *holds for any interval* $[\alpha, \beta] \subset [-1, 1]$.

Theorem A is a direct consequence of Theorem B, hence we need only prove Theorem B.

## 2. Applications

### 2.1. *Orthogonal Polynomials*

Let $\nu$ be a finite positive Borel measure on $[-1, 1]$. Then there exists a sequence of uniquely determined polynomials $q_n \in \Pi_n$,

$$q_n(x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

where

$$\int_{-1}^{1} q_n(x)\, q_m(x)\, d\nu(x) = \delta_{n,m}.$$

The zeros $x_i$, $1 \leqslant i \leqslant n$, of $q_n$ are all simple and located in $(-1, 1)$. Let $\tau_n$ be the zero counting measure of $q_n$. To obtain lower bounds for $|q_n'(x_i)|$ we follow a suggestion of P. Nevai [8]: The Christoffel–Darboux formula (cf. Szegő [10, p. 43]) yields

$$\sum_{k=0}^{n-1} q_k^2(x_i) = \frac{\gamma_{n-1}}{\gamma_n} q_n'(x_i)\, q_{n-1}(x_i)$$

and therefore

$$\begin{aligned} 2\gamma_0\, |q_{n-1}(x_i)| &= 2\, |q_0(x_i)\, q_{n-1}(x_i)| \\ &\leqslant q_0^2(x_i) + q_{n-1}^2(x_i) \\ &\leqslant \frac{\gamma_{n-1}}{\gamma_n} q_n'(x_i)\, q_{n-1}(x_i) \\ &\leqslant 2q_n'(x_i)\, q_{n-1}(x_i), \end{aligned}$$

since $\gamma_{n-1} \leqslant 2\gamma_{n-2}$ (cf. [5, p. 45]). Hence

$$\left|\frac{1}{\gamma_n} q_n'(x_i)\right| \geqslant \frac{\gamma_0}{\gamma_n}.$$

Let $T_n$ be the Chebyshev polynomial of degree $n$; then the minimum property of $T_n$ yields

$$\frac{\|q_n\|}{\gamma_n} \geqslant \|T_n\| = \frac{1}{2^{n-1}}$$

and therefore

$$\left|\frac{1}{\gamma_n} q_n'(x_i)\right| \geqslant \frac{\gamma_0}{\|q_n\|} \frac{1}{2^{n-1}}. \tag{2.1}$$

Moreover, the extremal property of the orthonormal polynomials $q_n$ leads to

$$1 = \|q_n\|_2 \leqslant \gamma_n \|T_n\|_2 \leqslant \frac{\gamma_n}{\gamma_0} \frac{1}{2^{n-1}},$$

where $\|\cdot\|_2$ is the $L^2$-norm with respect to the Borel measure $\nu$ on $[-1, 1]$. Consequently,

$$\frac{\|q_n\|}{\gamma_n} \leqslant \frac{\|q_n\|}{\gamma_0} \frac{1}{2^{n-1}}. \tag{2.2}$$

Hence we obtain from Theorem A

COROLLARY 1. *There exists a constant $c > 0$ such that the zero counting measure $\tau_n$ of the orthonormal polynomial $q_n$ satisfies*

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant c \frac{\log n}{n} (\log \|q_n\| + \log n)$$

*for any interval $[\alpha, \beta] \subset [-1, 1]$ and any $n \geqslant 2$.*

If $\nu' \geqslant \kappa > 0$ then Erdős and Turán [3] used the estimate

$$\|q_n\| = O(n)$$

to obtain

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant c \sqrt{\frac{\log n}{n}}.$$

In this case Corollary 1 yields

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant c \frac{(\log n)^2}{n}.$$

This estimate was announced by Erdős and Turán in [4, p. 111] without proof.

If we apply Corollary 1 to the zeros of the orthonormal Pollaczek polynomials (cf. Szegő [10, Appendix, p. 393]) then

$$|(\tau_n-\mu)([\alpha,\beta])|=O\left(\frac{\log n}{\sqrt{n}}\right),$$

since the Chebyshev norm of these polynomials is $O(\exp(\alpha n^{1/2}))$ for some constant $\alpha>0$.

## 2.2. *Chebyshev Approximation*

Let $f\in C[-1,1]$ and let $p_n^*$ denote the polynomial of best approximation to $f$ with respect to $\Pi_n$. Then there exist $n+2$ alternation points

$$-1\leqslant y_0<y_1<\cdots<y_{n+1}\leqslant 1 \tag{2.3}$$

of the error function $f-p_n^*$ such that

$$(f-p_n^*)(y_i)=\|f-p_n^*\|,\qquad 0\leqslant i\leqslant n+1, \tag{2.4}$$

and

$$|(f-p_n^*)(y_i)|=(-1)^i\,\delta(f-p_n^*)(y_i),\qquad 0\leqslant i\leqslant n+1, \tag{2.5}$$

where $\delta=1$ or $\delta=-1$ is fixed and $\|\cdot\|$ is the Chebyshev norm on $[-1,1]$. If we associate with $p_n$ the discrete measure

$$\mu_n(A):=\frac{\text{number of alternation points } y_i \text{ in } A}{n+2}, \tag{2.6}$$

where $A$ is any point set of $[-1,1]$, then Kadec [6] proved for any $\varepsilon>0$ that

$$|(\mu_n-\mu)([\alpha,\beta])|\leqslant c_\varepsilon\frac{1}{n^{1/2-\varepsilon}} \tag{2.7}$$

for a subsequence of integers $n$, where $c_\varepsilon$ is an absolute constant depending on $\varepsilon$.

In [1, 2] Blatt and Lorentz showed how to improve (2.7) by using the Erdős–Turán estimate (1.3): Let

$$p_n^*(z)=a_nz^n+\cdots,$$
$$e_n=\|f-p_n^*\|,$$

and

$$p_{n+1}^* - p_n^* = a_{n+1} T_{n+1} + q_n,$$

where $q_n \in \Pi_n$ and $T_{n+1}$ is the Chebyshev polynomial of degree $n+1$. Then

$$|a_{n+1}| \, \|T_{n+1}\| \leqslant e_n - e_{n+1}$$

and therefore

$$|a_{n+1}| \leqslant (e_n - e_{n+1}) 2^n.$$

Hence, if $a_{n+1} \neq 0$ the polynomial

$$p_{n+1} := \frac{p_{n+1}^* - p_n^*}{a_{n+1}} \tag{2.8}$$

is a monic polynomial and

$$\|p_{n+1}\| \leqslant \frac{e_n + e_{n+1}}{e_n - e_{n+1}} \frac{1}{2^n}. \tag{2.9}$$

Following the reasoning of Kadec, since $\lim_{n \to \infty} e_n = 0$ there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ such that $e_{n_1} \leqslant 1$ and

$$e_{n+1} \leqslant \left(1 - \frac{4}{n^2}\right) e_n \qquad \text{for} \quad n = n_j, j = 1, ...,$$

and therefore, for such $n$

$$\frac{e_n + e_{n+1}}{e_n - e_{n+1}} \leqslant \frac{n^2}{2} \tag{2.10}$$

or by (2.9)

$$\|p_{n+1}\| \leqslant n^2 \left(\frac{1}{2}\right)^{n+1}. \tag{2.11}$$

Now, the alternation points $x_i$ are separated by the zeros of $p_{n+1}$ and the Erdős–Turán estimate (1.3) yields

$$|(\mu_n - \mu)([\alpha, \beta])| \leqslant c \sqrt{\frac{\log n}{n}}$$

for the subsequence $n \in \{n_j\}_{j=1}^{\infty}$.

But it is also possible to obtain lower bounds for the modulus of $p'_{n+1}$ at the zeros: Because of (2.3)–(2.5)

$$(-1)^i \delta(p^*_{n+1} - p^*_n)(y_i) = (-1)^i \delta[(f - p^*_n)(y_i) - (f - p^*_{n+1})(y_i)]$$
$$\geqslant e_n - e_{n+1}$$

or, if $a_{n+1} \neq 0$

$$\operatorname{sign}(a_{n+1})(-1)^i \delta p_{n+1}(y_i) \geqslant \frac{e_n - e_{n+1}}{|a_{n+1}|}.$$

Since

$$|a_{n+1}| \, \|T_{n+1}\| \leqslant \|p^*_{n+1} - p^*_n\| \leqslant e_n + e_{n+1},$$

together with (2.10), we conclude that

$$\operatorname{sign}(a_{n+1})(-1)^i \delta p_{n+1}(y_i) \geqslant \frac{4}{n^2}\left(\frac{1}{2}\right)^{n+1} \tag{2.12}$$

for a subsequence of integers $n$. Let

$$x_i, \qquad i = 1, ..., n+1,$$

be the zeros of $p_{n+1}$; then

$$-1 \leqslant y_0 < x_0 < y_1 < x_1 < \cdots < y_n < x_n < y_{n+1} \leqslant 1 \tag{2.13}$$

and the inequality (2.12) leads to the crucial lower bound for $|p'_{n+1}(x_i)|$, namely

LEMMA 1. *For all* $0 \leqslant i \leqslant n$

$$|p'_{n+1}(x_i)| \geqslant \frac{2}{n^2}\left(\frac{1}{2}\right)^{n+1}. \tag{2.14}$$

Then Theorem A, together with the separating condition (2.13), yields

COROLLARY 2. *Let* $[\alpha, \beta] \subset [-1, 1]$. *Then the discrepancy between the equilibrium distribution* $\mu$ *and the measure* $\mu_n$, *counting the alternation points* $y_i$ *of* $f - p^*_n$, *can be estimated by*

$$|(\mu_n - \mu)([\alpha, \beta])| \leqslant c \frac{(\log n)^2}{n}$$

*for a subsequence of integers n, where c is an absolute constant independent of f and n.*

It remains to prove (2.14).

*Proof of Lemma* 1. We may confine ourselves to two situations:

$$(\alpha)\ 0<i<n, \qquad p_{n+1}(y_i)\leqslant\frac{-4}{n^2}\left(\frac{1}{2}\right)^{n+1}, \qquad p_{n+1}(y_{i+1})\geqslant\frac{4}{n^2}\left(\frac{1}{2}\right)^{n+1}:$$

Then let $\tilde{x}_i<x_i<\tilde{x}_{i+1}$ be the zeros of $p'_{n+1}$ nearest to $x_i$. Since all zeros of $p_{n+1}$ are real and simple the same property holds for all derivatives. Therefore

$$p_{n+1}(\tilde{x}_i)\leqslant\frac{-4}{n^2}\left(\frac{1}{2}\right)^{n+1} \tag{2.15}$$

and

$$p_{n+1}(\tilde{x}_{i+1})\geqslant\frac{4}{n^2}\left(\frac{1}{2}\right)^{n+1}. \tag{2.16}$$

Let us now assume that

$$0<p'_{n+1}(x_i)<\frac{2}{n^2}\left(\frac{1}{2}\right)^{n+1}. \tag{2.17}$$

Since $\tilde{x}_{i+1}-x_i<2$, there exists because of (2.15) and (2.16) a point $x_i<\eta_i<\tilde{x}_{i+1}$ such that $p'_{n+1}(\eta_i)>(2/n^2)(1/2)^{n+1}$. Consequently, there exists a point

$$x_i<\tilde{\eta}_i<\tilde{x}_{i+1} \qquad \text{such that} \quad p''_{n+1}(\tilde{\eta}_i)>0. \tag{2.18}$$

Analogously, there exists a point

$$\tilde{x}_i<\tilde{\tilde{\eta}}_i<x_i \qquad \text{such that} \quad p''_{n+1}(\tilde{\tilde{\eta}}_i)<0. \tag{2.19}$$

Since $p_{n+1}$ has a relative minimum at $\tilde{x}_i$ and a relative maximum at $\tilde{x}_{i+1}$, we know that

$$p''_{n+1}(\tilde{x}_i)>0 \qquad \text{and} \qquad p''_{n+1}(\tilde{x}_{i+1})<0. \tag{2.20}$$

Because of (2.18)–(2.20) there exist at least 3 zeros of $p''_{n+1}$ in the open interval $(\tilde{x}_i, \tilde{x}_{i+1})$. Moreover, there exist $i$ zeros of $p'_{n+1}$ in $(-\infty, \tilde{x}_i]$ and $n-i$ zeros of $p'_{n+1}$ in $[\tilde{x}_{i+1}, \infty)$. By Rolle's theorem, the function $p''_{n+1}$ has at least $i-1$ zeros in $(-\infty, \tilde{x}_i)$ and $n-i-1$ zeros in $(\tilde{x}_{i+1}, \infty)$. Adding together the number of all these zeros we have $n+1$ zeros which contradicts $p''_{n+1} \in \Pi_{n-1}$, $p''_{n+1} \neq 0$.

($\beta$) $i=0$ or $i=n$:
As in case ($\alpha$) an analogous argument concerning the zeros of $p''_{n+1}$ leads to a contradiction if we assume that

$$|p'_{n+1}(x_0)| < \frac{2}{n^2}\left(\frac{1}{2}\right)^{n+1} \qquad \text{or} \qquad |p'_{n+1}(x_{n+1})| < \frac{2}{n^2}\left(\frac{1}{2}\right)^{n-1}.$$

## 3. Proof of Theorem B

For $z \in \mathbb{C} \setminus [-1, 1]$ let

$$\varphi(z) = z + \sqrt{z^2 - 1},$$

where $\sqrt{z^2-1}$ is asymptotically $z$ near infinity. Then $\varphi(z)$ maps $\mathbb{C} \setminus [-1, 1]$ conformally to the exterior of the unit disk. The inequality (1.12), resp. (1.13), can be written as

$$|\mathrm{Re}\, h(z)| \leqslant \kappa \frac{\log C_n}{n} \tag{3.1}$$

for all $z$ with $G(z) \geqslant \log \sigma_n$, where

$$h(z) = \frac{1}{n} \log p_n(z) - \log \varphi(z) - \log \frac{1}{2}. \tag{3.2}$$

Differentiating we obtain for $z \notin [-1, 1]$

$$h'(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{z - x_i} - \frac{\varphi'(z)}{\varphi(z)}. \tag{3.3}$$

Let $p$ be any polynomial and let us consider the integral

$$I = \frac{1}{2\pi i} \int_{\Gamma_\sigma} h'(z)\, p(z)\, dz \tag{3.4}$$

over the ellipse $\Gamma_\sigma$ in the positive sense, $\sigma > 1$. The integral in (3.4) is independent of $\sigma$ and

$$I = \frac{1}{n} \sum_{i=1}^{n} p(x_i) - \frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{\varphi'(z)}{\varphi(z)}\, p(z)\, dz. \tag{3.5}$$

Now, we want to construct the polynomial $p$ as an approximation of the characteristic function of $[\alpha, \beta]$ to obtain by (3.5) an estimate of

$$|(\tau_n - \mu)([\alpha, \beta])|.$$

First remark that we have only to consider the case $[\alpha, \beta] = [\alpha, 1]$ and to prove

$$(\tau_n - \mu)([\alpha, 1]) \leqslant c \frac{\log C_n}{n} \log n, \tag{3.6}$$

because (3.6) implies analogously

$$(\tau_n - \mu)([-1, \alpha]) \leqslant c \frac{\log C_n}{n} \log n. \tag{3.7}$$

But then (3.6) and (3.7) yield

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant 2c \frac{\log C_n}{n} \log n$$

for any interval $[\alpha, \beta] \subset [-1, 1]$.

In the following let $\mu([\alpha, 1]) \leqslant 1 - 1/n$, since otherwise

$$(\tau_n - \mu)([\alpha, 1]) \leqslant \frac{1}{n}$$

and (3.6) is proved.

Fix $\gamma > 0$ such that $\alpha = \cos \gamma$ and $\gamma \in [0, \pi]$. Now, we construct an approximation $\chi_n(t)$ for the characteristic function $\chi(t)$ of the interval $[-\gamma, \gamma]$ as follows: Let

$$u(t) = \begin{cases} 0 & \text{for} \quad |t| \leqslant \gamma \text{ and } |t| \geqslant \gamma + 1/n \\ -n^2 & \text{for} \quad \gamma < |t| < \gamma + 1/2n \\ n^2 & \text{for} \quad \gamma + 1/2n < |t| < \gamma + 1/n \end{cases}$$

and define

$$\chi_n(t) = 4 \int_{-\infty}^{t} \int_{-\infty}^{x} u(\xi)\, d\xi\, dx.$$

Then $\chi_n'(t)$ is continuously differentiable and

$$\chi_n'(t) \leqslant 2n \qquad \text{for all} \quad t \in \mathbb{R}$$

and twice differentiable at all $t$ where

$$|t| \neq \left\{\gamma, \gamma+\frac{1}{2n}, \gamma+\frac{1}{n}\right\}$$

with

$$x_n''(t) \leqslant 4n^2. \tag{3.8}$$

Moreover, $0 \leqslant \chi_n(t) \leqslant 1$ and

$$\chi_n(t)=1 \qquad \text{for} \quad |t| \leqslant \gamma,$$
$$\chi_n(t)=0 \qquad \text{for} \quad |t| \geqslant \gamma+1/n.$$

By Jackson's theorem there exists an odd trigonometric polynomial

$$s_n(t)=\sum_{v=1}^{n^4} a_v \sin vt \tag{3.9}$$

of degree at most $n^4$ such that for all $t \in [-\pi, \pi]$

$$|\chi_n' - s_n(t)| \leqslant c_1 \omega\left(\frac{1}{n^4}\right) \leqslant c_2 \frac{1}{n^2}, \tag{3.10}$$

where $\omega(t)$, $t>0$, is the modulus of continuity of the function $\chi_n'(t)$ and $c_1$, $c_2>0$ are absolute constants. Next, we integrate $\chi_n' - s_n$ and obtain

$$\int_0^t (\chi_n' - s_n)(x)\, dx = \chi_n(t) - S_n(t),$$

where

$$S_n(t)= -\sum_{v=1}^{n^4} \frac{a_v}{v} \cos vt + b_0$$

and

$$b_0=\chi_n(0)+\sum_{v=1}^{n^4} \frac{a_v}{v}.$$

Because of (3.10) there exists a constant $c_3>0$, independent of $n$, such that

$$|\chi_n(t) - S_n(t)| \leqslant \frac{c_3}{n^2}$$

for all $t \in [-\pi, \pi]$. Now, let us define the polynomial $p$ arising in (3.4) by

$$p(z) = -\sum_{\nu=1}^{n^4} \frac{a_\nu}{\nu} T_\nu(z) + b_0 = -\frac{1}{2} \sum_{\nu=1}^{n^4} \frac{a_\nu}{\nu} \left(\omega^\nu + \frac{1}{\omega^\nu}\right) + b_0,$$

where $\omega = \varphi(z)$. We observe that

$$p(\cos t) \leqslant \begin{cases} 1 + (c_3/n^2) & \text{for all } t \\ c_3/n^2 & \text{for } |t| \geqslant \gamma + (1/n) \end{cases}$$

and

$$p(\cos t) \geqslant \begin{cases} 1 - (c_3/n^2) & \text{for all } |t| \leqslant \gamma \\ -c_3/n^2 & \text{for all } t. \end{cases}$$

Hence,

$$\frac{1}{n} \sum_{i=1}^{n} p(x_i) \geqslant \tau_n([\alpha, 1]) - \frac{c_3}{n^2}. \tag{3.11}$$

Let $z = \psi(w)$ be the inverse mapping of $\varphi(z)$; then

$$\frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{\varphi'(z)}{\varphi(z)} p(z)\, dz = \frac{1}{2\pi} \int_0^{2\pi} p(\psi(\sigma e^{it}))\, dt,$$

where $w = \sigma e^{it}$, $\sigma > 1$. Fix $\sigma^* > 1$. Since the above integrals are independent of $\Gamma_\sigma$, $\sigma > 1$, and the function $p \circ \psi$ is uniformly continuous on $\{w\colon 1 \leqslant |w| \leqslant \sigma^*\}$, then

$$\operatorname{Re}\left(\frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{\varphi'(z)}{\varphi(z)} p(z)\, dz\right) \leqslant \mu([\alpha, 1]) + \frac{c_4}{n}. \tag{3.12}$$

Inserting (3.11) and (3.12) in (3.5) we have

$$\operatorname{Re} I \geqslant (\tau_n - \mu)([\alpha, 1]) - \frac{c_5}{n}, \tag{3.13}$$

where $c_5$ is an absolute constant. On the other hand

$$I = \frac{1}{2\pi i} \int_{|w| = \sigma} \left(\frac{1}{n} \frac{p_n'(\psi(w))\, \psi'(w)}{p_n(\psi(w))} - \frac{1}{w}\right) p(\psi(w))\, dw$$

$$= \frac{1}{2\pi i} \frac{1}{n} \int_{|w| = \sigma} \frac{d}{dw} \left(\log \frac{p_n(\psi(w)) 2^n}{w^n}\right) p(\psi(w))\, dw$$

and partial integration yields

$$I = -\frac{1}{2\pi i}\frac{1}{n}\int_{|w|=\sigma} \log\frac{p_n(\psi(w))2^n}{w^n}\frac{d}{dw}\,p(\psi(w))\,dw. \tag{3.14}$$

The function

$$H(w) = \frac{1}{n}\log\frac{p_n(\psi(w))2^n}{w^n}$$

is a single-valued analytic function in $|w| > 1$, including the point at infinity if we fix $H(\infty) = 0$. Because of (1.12) we know that

$$|\mathrm{Re}\, H(w)| \leqslant \kappa\,\frac{\log C_n}{n}$$

for all $z$ with $|w| \geqslant \sigma_n$. Then there exists a constant $c_6 > 0$ such that

$$|\mathrm{Im}\, H(w)| \leqslant c_6\,\frac{\log n \log C_n}{n} \tag{3.15}$$

for all $|w| \geqslant 1 + 2n^{-8}$ (cf. Pólya and Szegő [9, Problem 288, p. 140]). The Laurent series of

$$H(w) = \sum_{k=1}^{\infty} c_k w^{-k}$$

has real coefficients and therefore, by (3.14),

$$I = \frac{1}{4\pi i}\int_{|w|=\sigma}\left(\sum_{k=1}^{\infty} c_k w^{-k}\right)\left(\sum_{\nu=1}^{n^4} a_\nu\left(w^\nu - \frac{1}{w^\nu}\right)\right)\frac{1}{w}\,dw = \frac{1}{2}\sum_{k=1}^{n^4} a_k c_k$$

is real-valued. On the other hand, for $w = \sigma e^{it}$ and

$$p^*(w) = \sum_{\nu=1}^{n^4} a_\nu w^\nu \tag{3.16}$$

we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} \mathrm{Im}(H(w))\,\mathrm{Im}(p^*(w))\,dt\\
&\quad = -\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty} c_k\sigma^{-k}\sin kt\right)\left(\sum_{\nu=1}^{n^4} a_\nu\sigma^\nu \sin \nu t\right)dt\\
&\quad = -\pi\sum_{k=1}^{n^4} a_k c_k.
\end{aligned}$$

Hence,

$$I = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Im}(H(w)) \operatorname{Im}(p^*(w))\, dt$$

where $w = \sigma e^{it}$, $\sigma > 1$. By (3.15), for $|w| = 1 + 2n^{-8}$

$$|I| \leqslant \frac{c_6}{2\pi} \frac{\log n \log C_n}{n} \int_{-\pi}^{\pi} |\operatorname{Im} p^*(w)|\, dt. \tag{3.17}$$

The following lemma shows that the last integral is bounded. Consequently, there exists $c_7 > 0$ such that

$$|I| \leqslant c_7 \frac{\log C_n}{n} \log n,$$

so (3.13) leads to

$$(\tau_n - \mu)([\alpha, 1]) \leqslant c_8 \frac{\log C_n}{n} \log n,$$

with absolute constant $c_8$, independent of $n$.

It remains to estimate the integral in (3.17).

LEMMA 2. *Let $w = \sigma e^{it}$, $\sigma = 1 + 2n^{-8}$; then the integral*

$$\int_{-\pi}^{\pi} |\operatorname{Im} p^*(w)|\, dt$$

*is bounded by a fixed constant, independent of $n$.*

*Proof.* We obtain for $w = e^{it}$

$$p^*(w) = \tilde{s}_n(t) + i s_n(t),$$

where $s_n(t)$ is defined by (3.9) and $\tilde{s}_n(t)$ is its conjugate trigonometric polynomial. We have

$$\begin{aligned}
\int_{-\pi}^{\pi} |s_n(t)|^2\, dt &\leqslant \int_{-\pi}^{\pi} \left(\chi_n'(t) + \frac{c_2}{n^2}\right)^2 dt \\
&\leqslant \int_{-\pi}^{\pi} |\chi_n'(t)|^2\, dt + \frac{2c_2}{n^2} \int_{-\pi}^{\pi} |\chi_n'(t)|\, dt + \frac{c_2^2}{n^4} 2\pi \\
&\leqslant 4^3 n^4 \int_{-\gamma - 1/n}^{-\gamma - 1/2n} \left(t + \gamma + \frac{1}{n}\right)^2 dt + \frac{4c_2}{n^2} + \frac{c_2^2}{n^4} 2\pi \\
&\leqslant \frac{8}{3} n + \frac{4c_2}{n^2} + \frac{c_2^2}{n^4} \leqslant c_9 n
\end{aligned}$$

for all $n \geqslant 1$, where $c_9$ is a constant which we choose greater than 4 for later purposes. Therefore

$$\int_{-\pi}^{\pi} |\tilde{s}_n(t)|^2 \, dt = \int_{-\pi}^{\pi} |s_n(t)|^2 \, dt \leqslant c_9 n. \tag{3.18}$$

Now, we assert

$$|\tilde{s}_n(t)| \leqslant c_9 n^3 \qquad \text{for all } t. \tag{3.19}$$

Let us assume that (3.19) is false. Then there exists a point $t_0$ such that

$$|\tilde{s}_n(t_0)| = \max_{-\pi \leqslant t \leqslant \pi} |\tilde{s}_n(t)| \geqslant c_9 n^3.$$

Bernstein's inequality yields

$$|\tilde{s}_n'(t)| \leqslant n^4 \max_{-\pi \leqslant t \leqslant \pi} |\tilde{s}_n(t)| = n^4 \, |\tilde{s}_n(t_0)|$$

for all $t$ and by the mean value theorem

$$|\tilde{s}_n(t) - \tilde{s}_n(t_0)| \leqslant n^4 \, |\tilde{s}_n(t_0)| \, |t - t_0|$$

it follows that

$$|\tilde{s}_n(t)| \geqslant |\tilde{s}_n(t_0)| \, (1 - n^4 \, |t - t_0|) \geqslant \frac{c_9}{2} n^3$$

for all $|t - t_0| < 1/2n^4$ and therefore

$$\int_{-\pi}^{\pi} |\tilde{s}_n(t)|^2 \, dt \geqslant \frac{c_9^2}{4} n^2 > c_9 n$$

because of $c_9 > 4$. But this is in contrast to (3.18).

For $w = e^{it}$ we obtain from (3.10) and (3.19)

$$|p^*(w)| = \sqrt{|s_n(t)|^2 + |\tilde{s}_n(t)|^2} \leqslant c_{10} n^3.$$

An inequality of F. Riesz (cf. [7, p. 40]) yields

$$\left| \frac{d}{dw} p^*(w) \right| \leqslant c_{10} n^7$$

for all $|w| = 1$ and therefore

$$\left| \frac{d}{dw} p^*(w) \right| \leqslant c_{10} n^7 \sigma^{n^4} \leqslant c_{11} n^7$$

for $|w| \leqslant \sigma = 1 + 2n^{-8}$. Then

$$|p^*(w) - p^*(w_0)| \leqslant \frac{2c_{11}}{n}$$

for all $w = w(t) = (1 + 2n^{-8})e^{it}$, $w_0 = w_0(t) = e^{it}$. Finally, we have

$$\int_{-\pi}^{\pi} |\mathrm{Im}\, p^*(w)|\, dt \leqslant \int_{-\pi}^{\pi} |\mathrm{Im}\, p^*(w) - \mathrm{Im}\, p^*(w_0)|\, dt + \int_{-\pi}^{\pi} |\mathrm{Im}\, p^*(w_0)|\, dt$$

$$\leqslant \frac{4\pi c_{11}}{n} + \frac{2\pi c_2}{n^2} + \int_{-\pi}^{\pi} |\chi_n'(t)|\, dt.$$

But then the right-hand side is bounded since

$$\int_{-\pi}^{\pi} |\chi_n'(t)|\, dt \leqslant 2.$$

## 4. Simple Zeros on the Unit Circle

In [4] Erdős and Turán investigated the distribution of zeros of a monic polynomial $p_n \in \Pi_n$ bounded on the unit disk. Let us now assume in this case that all zeros $z_i$ of $p_n$ are simple zeros of the unit circle. Then we have to replace (1.4) by

$$\max_{|z| \leqslant 1} |p_n(z)| \leqslant A_n \tag{4.1}$$

and the inequality (1.6) by

$$|p_n'(z_i)| \geqslant \frac{1}{B_n}, \tag{4.2}$$

observing that the capacity 1 of the unit disk takes over the role of the capacity of $[-1, 1]$, which is $\frac{1}{2}$.

If we reformulate conditions (4.1) and (4.2) using the logarithmic potentials of $\tau_n$ and the arclength measure $\mu$ of the unit circle, we have to substitute these inequalities by

$$|U^{\tau_n}(z) - U^{\mu}(z)| \leqslant \kappa \frac{\log C_n}{n} \tag{4.3}$$

for all $|z| \geqslant 1+n^{-8}$. We remark that (4.3) is just the same inequality as (1.10) of Theorem B since

$$U^{\mu}(z) = G(z) = \log |z|.$$

Now, some slight modifications in the above proofs immediately yield

THEOREM C. *Let $p_n \in \Pi_n$, $n \geqslant 2$, be a monic polynomial with simple zeros on the unit circle such that either* (4.1) *and* (4.2) *or* (4.3) *hold. Then for any subarc*

$$S_{\alpha,\beta} = \{z\colon |z| = 1, \alpha \leqslant \arg z \leqslant \beta\} \qquad (\alpha \leqslant \beta)$$

*of the unit circle,*

$$|(\tau_n - \mu)(S_{\alpha,\beta})| \leqslant c \frac{\log C_n}{n} \log n, \tag{4.4}$$

*where c is an absolute constant independent of n and* $C_n = \max(A_n, B_n, n)$ *in the case* (4.1), (4.2).

## 5. How sharp Are the Results?

Let $T_n$ be the Chebyshev polynomial of degree $n$. Then $T_n(x)$ has the zeros

$$\xi_j = \cos \frac{2j-1}{n} \frac{\pi}{2}, \qquad 1 \leqslant j \leqslant n,$$

and the zero counting measure $\tau_n$ satisfies

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant \frac{1}{n}$$

for any interval $[\alpha, \beta] \subset [-1, 1]$. Since

$$\|T_n\| = \frac{1}{2^{n-1}} \qquad \text{and} \qquad |T_n'(\xi_j)| = \frac{1}{2^{n-1}} \left| \frac{n}{\sin(2j-1)(\pi/2n)} \right| \geqslant \frac{n}{2^{n-1}},$$

Theorem A yields

$$|(\tau_n - \mu)([\alpha, \beta])| \leqslant c \frac{(\log n)^2}{n}.$$

If we consider the same polynomials on the interval $I = [-1, 1+(\log n/n)^2]$, then some calculations together with Theorem A show

$$|(\tau_n - \tilde{\mu})([\alpha, \beta])| \leqslant c\,\frac{(\log n)^2}{n}$$

for any subinterval $[\alpha, \beta]$ of $I$, where $\tilde{\mu}$ denotes the equilibrium distribution of $I$. But the real discrepancy between $\tau_n$ and $\tilde{\mu}$ is $O(\log n/n)$. Hence Theorem A seems not far way from being optimal.

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